Multipurpose Optimization with and without Uncertainties for Deformed Bodies and Structural Elements

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Introduction

OUR TEAM - Laboratory of Mechanics and Structural Optimization

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Multipurpose Optimization with and without Uncertainties for Deformed Bodies and Structural Elements
Pareto approch

\[ J(h) = \{ J_1(h), \ldots, J_i(h), \ldots, J_N(h) \}^T \rightarrow \min_{h \in \Lambda} \]

\[ h = \arg\min_{h \in \Lambda} J(h) \]

\[ J_C(h) = \sum C_i J_i(h) \rightarrow \min_{h \in \Lambda} \]

\[ C_i \geq 0, \sum C_i = 1, i = 1, \ldots, N \]
Nash Approach

\[ J_1(h_1, h_2), J_2(h_1, h_2), h_1 \in \Lambda_{h_1}, h_2 \in \Lambda_{h_2} \]

Suppose \( h_2 \) given

\[ J_1(h_1, h_2) \rightarrow \min_{h_1 \in \Lambda_{h_1}} \]

Optimal solution

\[ h_1 = \arg\min_{h_1 \in \Lambda_{h_1}} J_1(h_1, h_2) \]

Suppose \( h_1 \) given

\[ J_2(h_1, h_2) \rightarrow \min_{h_2 \in \Lambda_{h_2}} \]

Optimal solution

\[ h_2 = \arg\min_{h_2 \in \Lambda_{h_2}} J_1(h_1, h_2) \]
Uncertainty in the problems of multi-objective optimization:

1. Guaranteed approach (minimax) - implementation scenario "is based on worst case";

2. Probabilistic approach (stochastic) - applicable when available to the necessary statistical data;

Multi-layered nonhomogeneous plate and striker

\[ x_i \leq x \leq x_{i+1} = x_i + \Delta_{i+1}, \quad x_0 = 0, \quad x_n = L \]
THE PROBLEM OF SUPERSONIC STRIKERS PENETRATION INTO DEFORMABLE MEDIA

Analysis

Statement of the problem

Discrete family of given materials

Piece-wise constant functions:

\[ A_0(x) \] – Dynamical hardness
\[ A_2(x) \] – Density

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DYNAMICAL HIGH SPEED PENETRATION OF RIGID AXISYMMETRIC STRIKER

Statement of the problem

Resistance force

\[ D(x) = \pi R^2 \left( A_0(x) + \kappa A_2(x) v^2(x) \right) \]

\[ Mv \frac{dv}{dx} = -D(x), \quad v(0) = v_0 = v_{imp} \]
DYNAMICAL HIGH SPEED PENETRATION OF RIGID AXISYMMETRIC STRIKER IN GENERAL CASE

**Analysis**

Statement of the problem

Resistance force in general case

\[
D(x) = D_{nose}(x) + D_{lat}(x) = B_0(x) + B_2(x) v^2
\]

\[
B_0(x) = \pi r^2 A_0(x) - 2\pi \int_{x_*}^{x_{**}} A_0(\eta) y y_\eta d\eta
\]

\[
B_2(x) = \pi r^2 A_0(x) - 2\pi \int_{x_*}^{x_{**}} \frac{A_2(\eta) y y_\eta^3}{1 + y_\eta^2} d\eta
\]

\[
x_* = 0, x_{**} = x, \quad \text{when} \quad 0 \leq x < l
\]

\[
x_* = x - l, x_{**} = x, \quad \text{when} \quad l \leq x \leq L
\]

\[
x_* = x - l, x_{**} = L, \quad \text{when} \quad L < x \leq L + l
\]
Ballistic Limit Velocity

\[ v_{BLV} : \]

if \( v_0 = v_{imp} \) is such that

\[ v_n = (v)_{x=L} = 0, \]

then \( v(0) = v_{imp} = v_{BLV} \)
Spaced nodes used for solution of the initial-value problem

\[ \xi = L - x, \quad d \xi = -dx \]
BASIC RELATIONS

\[
\frac{dv^2}{d\xi} = \beta^{j+1} (\alpha^{j+1} + v^2), \quad v^2_0 = \left( v^2 \right)_{\xi=\xi_0} = 0
\]

\[
v_j = v(\xi_j), \quad \xi_{j+1} = \xi_j + \Delta_{j+1}, \quad j = 0, 1, 2, \ldots, n-1
\]

\[
\alpha^{j+1} = \frac{1}{\kappa} \left( \frac{A_0}{A_2} \right)^{j+1}
\]

\[
\beta^{j+1} = \frac{2\pi R^2 \kappa}{M} \left( A_2 \right)^{j+1}
\]

\[
v^2_n = \left( v^2 \right)_{\xi=\xi_n=L} = v^2_{BLV}
\]
EVALUATION OF BALLISTIC LIMIT VELOCITY

Analysis

Integration w.r.t. $\xi$

$$\ln\left(\frac{\alpha^{j+1} + v_{j+1}^2}{\alpha^{j+1} + v_j^2}\right) = \mu^{j+1}, \quad \xi_j \leq \xi \leq \xi_{j+1}$$

where $\mu^{j+1} = \beta^{j+1} \Delta_{j+1}$,

$$\Delta_{j+1} = \xi_{j+1} - \xi_j, \quad j = 0, 1, 2, ..., n - 1$$

Relation between $v_j^2$ and $v_{j+1}^2$

$$\frac{v_{j+1}^2}{\alpha^{j+1}} = \exp\left(\mu^{j+1}\right) - 1 + \frac{v_j^2}{\alpha^{j+1}} \exp\left(\mu^{j+1}\right)$$

$v_0 = 0$, \quad $v_n = v_{BLV}$

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Multipurpose Optimization with and without Uncertainties for Deformed Bodies and Structural Elements
MULTIOBJECTIVE OPTIMIZATION PROBLEM

v-blv - ballistic velocity, delta – variable depended on the solution

Analysis

Basic relations

\[ J_\ast = J(h) = \min_{h \in \Lambda_h} \left\{ \begin{array}{c} J_v \\ J_M \\ J_L \end{array} \right\} \]

\[ J_v = -v_{BLV}(t, \Delta) \]

\[ \Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n) \]

\[ J_M = S \int_0^L k(t(x)) \, dx \quad \text{- total mass} \]

\[ J_L = L = \Delta_1 + \Delta_2 + \ldots + \Delta_n \quad \text{- total thickness} \]
DESIGN VARIABLES

\[ h = \left( t, \Delta \right) = \left( t, \Delta_1, \Delta_2, \ldots, \Delta_n \right) \]

The set of admissible design variables:

\[ \Lambda_h = \left\{ \begin{array}{l}
 t : A_0 \left( t \left( x \right) \right) = A_0^{i+1}; \quad A_2 \left( t \left( x \right) \right) = A_2^{i+1}; \quad x \in \left[ x_i, x_{i+1} \right) \\
 i = 0, 1, 2, \ldots, n - 1; \quad \left( A_0^{i+1}, A_2^{i+1} \right) \in \left\{ \left( A_0 \right)_s, \left( A_2 \right)_s \right\} \\
 s = 1, 2, \ldots, r; \quad \Delta : \Delta_j > 0; \quad j = 1, 2, \ldots, n
\end{array} \right. \]
$h_* = \arg \min_{h \in \Lambda} J(h)$

$h_*$ is optimal if there is no other such that $\tilde{h}$

$J_i(\tilde{h}) \leq J_i(h_*), i = \mu, M, L$

and most for one criteria component

$J_i(\tilde{h}) < J_j(h_*)$
METHOD OF OBJECTIVE WEIGHTING

Preference functional:

\[ J_C = C_v J_v + C_M J_M + C_L J_L \]

\[ C_v \geq 0, \ C_M \geq 0, \ C_L \geq 0, \ C_v + C_M + C_L = 1 \]

\[ J_C (h_\star) = \arg \min_{h \in \Lambda_h} J_C (h) \]
OPTIMAL DESIGN BY GENETIC ALGORITHM $C_v > 0$ summ $cv=1$

\[ \Delta_1 = \Delta_2 = \ldots = \Delta_n = \frac{L}{n} \quad (n = 20) \]

\[ J = \left\{ \begin{array}{c} J_v \\ J_M \end{array} \right\} \rightarrow \min_{t \in \Lambda_t} \]

\[ J_C = C_v J_v + C_M J_M = \left(1 - C_M\right) J_v + C_M J_M \rightarrow \min_{t \in \Lambda_t} \]
### Materials:

<table>
<thead>
<tr>
<th>Material</th>
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<td>aluminum</td>
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Dependence of the striker velocity on the depth of penetration when $C_M=0$
### OPTIMAL DESIGN BY GENETIC ALGORITHM
(dependence on different values of functional weights)

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<td>2) $C_M = 0.3$</td>
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<td>4) $C_M = 0.5$</td>
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<td>5) $C_M = 0.55$</td>
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OPTIMAL DESIGN BY GENETIC ALGORITHM

Materials distributions:

6) $C_M = 0.57$
7) $C_M = 0.58$
8) $C_M = 0.59$
9) $C_M = 0.6$
10) $C_M = 0.605$
Monotonic dependence of \((-J_c)\) functional on parameter \(C_M\)
Dependence of \((-J_C)\) functional on generation number for different values of parameter \(C_M\)
Dependence of the striker velocity on the depth of penetration when $C_M=0$
Optimal layered structures are found in frame of multiobjective problem statement.

Global optimal structures are defined taken into account discrete set of available materials.

Defined optimal layered structures provide maximal ballistic limit velocity (BLV) of rigid strikers and have minimal total mass.

To solve this nonlocal optimization problem the numerical method of genetic algorithm is effectively used.

Entrance and exit effects for penetration in each layer of optimizing structure are ignored in this investigation.
Contact Optimization Problems with Uncertainties

GEOMETRY OF THE CONTACT INTERACTION

Contact interaction under external loading

\[ \Omega = \Omega_f + \Omega_q + \Omega_0 \]

\[ \Omega_q = \bigcup_{i=1}^{i=N} \Omega_q^i \]

\[ \Omega_f \mid \Omega_0 = 0, \quad \Omega_f \mid \Omega_q^i = 0, \]

\[ \Omega_q^i \mid \Omega_0 = 0 \]
STATEMENT OF THE PROBLEM

Design variable and external forces

\[ z = f(x, y), \quad (x, y) \in \Omega_f; \quad z = 0, \quad (x, y) \in \partial \Omega_f \]

\[ q^i_j = q^i_j(x, y, \xi, \eta), \quad (x, y) \in \Omega^i_q \]

\[ j = x, y, z; \quad i = 1, 2, \ldots, N \]
STATEMENT OF THE PROBLEM

Boundary conditions and reactions

\[ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0, \quad (x, y) \in \Omega_0 \]
\[ \sigma_{xz} = q_x^i(x, y, \xi, \eta), \quad \sigma_{yz} = q_y^i(x, y, \xi, \eta), \quad \sigma_{zz} = q_z^i(x, y, \xi, \eta), \quad (x, y) \in \Omega_q^i \]
\[ P(\xi, \eta) = \int_{\Omega_f} p(x, y, \xi, \eta) d\Omega_f, \]
\[ M_x(\xi, \eta) = \int_{\Omega_f} y p(x, y, \xi, \eta) d\Omega_f, \quad M_y(\xi, \eta) = \int_{\Omega_f} x p(x, y, \xi, \eta) d\Omega_f \]
STANDARD PUNCH AND ACTUAL PUNCH

**Standard punch with plane bottom**

\[ z = f^0(x, y) = \alpha + \beta x + \gamma y \quad (x, y) \in \Omega_f \]

\[ \alpha, \beta, \gamma - \text{given coefficients} \]

\[ q_x(x, y) = q_y(x, y) = q_z(x, y) = 0, \quad \quad (x, y) \in \Omega_q \]

\[ p^0(x, y) = \alpha p^0_\alpha(x, y) + \beta p^0_\beta(x, y) + \gamma p^0_\gamma(x, y), \quad \quad (x, y) \in \Omega_f \]

\[ u^0(x, y) = \alpha u^0_\alpha(x, y) + \beta u^0_\beta(x, y) + \gamma u^0_\gamma(x, y), \quad \quad (x, y) \in \Omega_f \]

\[ v^0(x, y) = \alpha v^0_\alpha(x, y) + \beta v^0_\beta(x, y) + \gamma v^0_\gamma(x, y), \quad \quad (x, y) \in \Omega_f \]

\[ w^0(x, y) = \alpha w^0_\alpha(x, y) + \beta w^0_\beta(x, y) + \gamma w^0_\gamma(x, y), \quad \quad (x, y) \in \Omega_q^i \]

**Actual punch**

\[ f = f(x, y) \]

\[ \sigma_{xz} = \sigma_{yz} = 0, \quad w = f(x, y) \]

\[ (x, y) \in \Omega_f \]

\[ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0, \quad (x, y) \in \Omega_f \]

\[ \sigma_{xz} = q^i_x(x, y, \xi, \eta), \quad \quad \sigma_{yz} = q^i_y(x, y, \xi, \eta), \quad \quad \sigma_{zz} = q^i_z(x, y, \xi, \eta), \quad \quad (x, y) \in \Omega_q^i \]
RECIROCITY RELATION

The first system

\[ w^0 = f^0(x, y), \quad p^0 \]

\[ (x, y) \in \Omega_f \]

\[ u^0, v^0, w^0 \quad (x, y) \in \Omega^i_q \]

\[ i = 1, 2, \ldots, N \]

The second system

\[ w = f(x, y), \quad p \]

\[ (x, y) \in \Omega_f \]

\[ q^i_x, q^i_y, q^i_z \quad (x, y) \in \Omega^i_q \]

\[ \int_{\Omega_f} p^0 f d\Omega_f = \int_{\Omega_f} f^0 p d\Omega_f + \]

\[ + \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0 q^i_x + v^0 q^i_y + w^0 q^i_z \right) d\Omega^i_q \]
**RECIPROCITY THEOREM**

\[
\alpha \left[ \int_{\Omega_f} pd\Omega_f - \int_{\Omega_f} p^0_{\alpha} f d\Omega_f + \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\alpha} q_x^i + v^0_{\alpha} q_y^i + w^0_{\alpha} q_z^i \right) d\Omega^i_q \right] + \\
+ \beta \left[ \int_{\Omega_f} xpd\Omega_f - \int_{\Omega_f} p^0_{\beta} f d\Omega_f + \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\beta} q_x^i + v^0_{\beta} q_y^i + w^0_{\beta} q_z^i \right) d\Omega^i_q \right] + \\
+ \gamma \left[ \int_{\Omega_f} ypd\Omega_f - \int_{\Omega_f} p^0_{\gamma} f d\Omega_f + \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\gamma} q_x^i + v^0_{\gamma} q_y^i + w^0_{\gamma} q_z^i \right) d\Omega^i_q \right] = 0
\]

\( \alpha, \beta, \gamma \) – arbitrary constants

\[ P = P_f (f) - P_q (q) \]

\[ M_j = M^f_j (f) - M^q_j (q), \quad j = x, y \]

\[ P_f (f) = \int_{\Omega_f} p^0_{\alpha} f d\Omega_f \]

\[ M^f_x (f) = \int_{\Omega_f} p^0_{\gamma} f d\Omega_f, \quad M^f_y (f) = \int_{\Omega_f} p^0_{\beta} f d\Omega_f \]

\[ P_q (q) = \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\alpha} q_x^i + v^0_{\alpha} q_y^i + w^0_{\alpha} q_z^i \right) d\Omega^i_q \]

\[ M^q_x (q) = \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\gamma} q_x^i + v^0_{\gamma} q_y^i + w^0_{\gamma} q_z^i \right) d\Omega^i_q \]

\[ M^q_y (q) = \sum_{i=1}^{N} \int_{\Omega^i_q} \left( u^0_{\beta} q_x^i + v^0_{\beta} q_y^i + w^0_{\beta} q_z^i \right) d\Omega^i_q \]
PROBABILISTIC APPROACH

Probability densities and joint probability distribution functions

\[ g^i(\xi, \eta) = \frac{\partial^2 F^i}{\partial \xi \partial \eta} \]

\[ g^i(\xi, \eta) = \frac{1}{\pi} \left\{ \begin{array}{l} 0, \quad (\xi, \eta) \notin \Omega_q \\ g^i(\xi, \eta), \quad (\xi, \eta) \in \Omega_q \end{array} \right\} \]

\[ \int_{\Omega_q} g^i(\xi, \eta) d\Omega_q = \pi \]
RANDOM LOAD

\[ q_j = q(x, y, \xi, \eta), \quad j = x, y, z \]

\[ \Omega_q \{ x_1 \leq x \leq x_2, y_1 \leq y \leq y_2 \} \text{ – the only domain} \]

\[ g = g(\xi, \eta) = g_\xi(\xi) g_\eta(\eta) \]

\[ g_\xi(\xi) = \begin{cases} 
0, & \xi < x_1, \\
\frac{1}{x_2 - x_1}, & x_1 < \xi < x_2, \\
0, & \xi > x_2, 
\end{cases} \]

\[ g_\eta(\eta) = \begin{cases} 
0, & \eta < y_1, \\
\frac{1}{y_2 - y_1}, & y_1 < \eta < y_2, \\
0, & \eta > y_2 
\end{cases} \]

\[ F_\xi(\xi) = \begin{cases} 
0, & \xi < x_1, \\
\frac{\xi - x_1}{x_2 - x_1}, & x_1 < \xi < x_2, \\
1, & \xi > x_2, 
\end{cases} \]

\[ F_\eta(\eta) = \begin{cases} 
0, & \eta < y_1, \\
\frac{\eta - y_1}{y_2 - y_1}, & y_1 < \eta < y_2, \\
1, & \eta > y_2 
\end{cases} \]

Mathematical expectations \( \hat{P}, \hat{M}_x, \hat{M}_y \)

\[ \hat{P} = \mathbb{E}(P) = P_f(f) - \mathbb{E}(P_q(q)), \quad M_j^f(f) - \mathbb{E}(M_j^q(q)), \quad j = x, y \]
FORMULATION OF OPTIMIZATION PROBLEM

Minimizing functional and constraints

\[ J = \rho \int_{\Omega_f} \sqrt{1 + (\nabla f)^2} \, d\Omega_f \approx \]
\[ \approx \rho S_f + \frac{\rho}{2} \int_{\Omega_f} (\nabla f)^2 \, d\Omega_f \rightarrow \min_f \]
\[ \hat{P} = P^*, \quad \hat{M}_x = M^*_x, \quad \hat{M}_y = M^*_y \]

\[ P^*, M^*_x, M^*_y - \text{given problem parameters} \]
\[ \Omega_f : \quad x^2 + y^2 \leq a^2 \]

Lagrange augmented functional

\[ J^L = \int_{\Omega_f} \left[ \frac{\rho}{2} (\nabla f)^2 - \lambda_\alpha p^0_\alpha f - \lambda_\beta p^0_\beta f - \lambda_\gamma p^0_\gamma f \right] \, d\Omega_f \]

\[ p^0_\alpha (r) = \frac{E}{\pi (1-v^2) \sqrt{a^2 - r^2}} \]
\[ p^0_\beta (r, \theta) = \frac{2Er \cos \theta}{\pi (1-v^2) \sqrt{a^2 - r^2}} \]
\[ p^0_\gamma (r, \theta) = \frac{2Er \sin \theta}{\pi (1-v^2) \sqrt{a^2 - r^2}} \]
\[ r = \sqrt{x^2 + y^2} \]
SOLUTION OF OPTIMIZATION PROBLEM

Analysis

Optimality condition and boundary condition

\[ \Delta \equiv \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial f}{\partial \theta^2} = \]

\[ = -\frac{\lambda_\alpha}{\rho} p^0_\alpha(r) - \frac{\lambda_\beta}{\rho} p^0_\beta(r, \theta) - \frac{\lambda_\gamma}{\rho} p_\gamma(r, \theta) \]

\[ (f(r, \theta))_{r=a} = 0, \quad 0 \leq \theta \leq 2\pi \]

Bounded solution and shape function

\[ f(r, \theta) = f_\alpha(r) + f_\beta(r, \theta) + f_\gamma(r, \theta), \]

\[ f_\alpha(r) = \lambda_\alpha \chi_\alpha(r), \]

\[ f_\beta(r, \theta) = \lambda_\beta \chi_\beta(r, \theta), \]

\[ f_\gamma(r, \theta) = \lambda_\gamma \chi_\gamma(r, \theta) \]
Some examples

SHAPE FUNCTIONS

Symmetric shape function

\[ \chi_\alpha(r) = \frac{E}{\rho \pi (1 - \nu^2)} \left( \sqrt{a^2 - r^2} - a \ln \left( \frac{a + \sqrt{a^2 - r^2}}{a} \right) \right) \]
ANALYTICAL SOLUTION FOR $i=\beta, \gamma$

Finding of nonsymmetric shape functions

$$\frac{\partial^2 \chi_\beta}{\partial r^2} + \frac{1}{r} \frac{\partial \chi_\beta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi_\beta}{\partial \theta^2} = -\frac{2}{\rho} r p_\alpha^0(r) \cos \theta$$

$$\chi_\beta(r, \theta) = \chi^0(r) \cos \theta$$

$$\frac{d^2 \chi^0}{dr^2} + \frac{1}{r} \frac{d \chi^0}{dr} - \frac{1}{r^2} \chi^0 = \frac{2K}{\rho} \frac{d}{dr} \left( \sqrt{a^2 - r^2} \right), \quad K = \frac{E}{\pi(1-\nu^2)}$$

$$r \chi^0(r) = -\frac{2K}{3\rho} \left( a^2 - r^2 \right)^{\frac{3}{2}} + \frac{C}{2} r^2 + D$$

$$C = -\frac{4K}{3\rho} a, \quad D = \frac{2Ka^3}{3\rho}$$
SHAPE FUNCTIONS

Nonsymmetric shape function for $i = \beta$

$$\chi_\beta(r, \theta) = \chi^0(r) \cos \theta = \frac{2E \cos \theta}{3\pi(1-\nu^2) \rho r} \left\{ a \left( a^2 - r^2 \right) - \left( a^2 - r^2 \right)^{3/2} \right\}$$
Some examples

SHAPE FUNCTIONS

Nonsymmetric shape function for \( i=\gamma \)

\[
\chi_{\gamma}(r, \theta) = \chi^0(r)\sin \theta = \frac{2E \sin \theta}{3\pi(1-\nu^2)} \rho r \left\{ a(a^2 - r^2) - \left( a^2 - r^2 \right)^{\frac{3}{2}} \right\}
\]
OPTIMAL PUNCH SHAPES

Some examples

Uniform distribution of probability density

\[
\frac{\mathcal{M}_x}{\mathcal{M}_y} = K_f, \quad K = \frac{E}{\pi (1 - v^2)}, \quad P^* = 1, \quad M^*_x = M^*_y = 0.25
\]

\[
\lambda_\alpha = \frac{K^2}{\rho} \lambda_\alpha \approx 1.7148
\]

\[
\lambda_\beta = \frac{K^2}{\rho} \lambda_\beta \approx 0.5810
\]

\[
\lambda_\gamma = \frac{K^2}{\rho} \lambda_\gamma \approx 0.4796
\]

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Some examples

OPTIMAL PUNCH SHAPES

Gauss distribution of probability density \((x_0=3, y_0=0)\)

\[
\begin{align*}
\mathcal{O}_\alpha &= \frac{K^2}{\rho} \lambda_\alpha \approx 1.7067 \\
\mathcal{O}_\beta &= \frac{K^2}{\rho} \lambda_\beta \approx 0.5745 \\
\mathcal{O}_\gamma &= \frac{K^2}{\rho} \lambda_\gamma \approx 0.4796 \\
\mathcal{O}_\text{max} &= 0.58
\end{align*}
\]

\[
f_\alpha = Kf, \quad K = \frac{E}{\pi(1-\nu^2)} \quad , \quad P^* = 1 \quad , \quad M_x^* = M_y^* = 0.25
\]

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Multipurpose Optimization with and without Uncertainties for Deformed Bodies and Structural Elements
SOME CONCLUSIONS

- Contact interaction of rigid punch and elastic foundation has been investigated under conditions of incomplete information regarding the external actions.

- Shape optimization problem with uncertainties has been formulated and studied in frame of probabilistic approach.

- With application of Betti’s reciprocity theorem the original probabilistic optimization problem has been simplified.

- Optimal shapes of the punches have been found analytically for different cases of the problem formulation.

- Some peculiarities of the optimized model have been studied.
Multipurpose Optimization with and without Uncertainties

Mixed boundary value problem

\[ \Omega = \Omega_f + \Omega_0 + \Omega_q \]

\( \Omega_0 : \sigma_n = \sigma_\tau = 0 \)

\( \Omega_q : \sigma_n = q_n, \quad \sigma_\tau = q_\tau \)

\( \Omega_f : w = f(x, y), \quad \sigma_\tau = \tau_0 + \mu p \)

Contact interaction of a moving rigid punch and elastic half-space

Deformable environ

Mixed boundary value problem with uncertaine boundaries

Variational problems with unilateral constraints
FUNCTIONAL CHARACTERISTICS

\[ P_\ast = \int_{\Omega_f} p(x,y) d\Omega_f, \]
\[ M_\ast = \int_{\Omega_f} xp(x,y) d\Omega_f, \]
\[ p \in \Lambda_p = \{ p(x,y) \geq 0, \int_{\Omega_f} p(x,y) d\Omega_f = P_\ast, \int_{\Omega_f} xp(x,y) d\Omega_f = M_\ast \}, P_\ast > 0, M_\ast > 0 \]
\[ J_F = \int_{\Omega_f} (\tau_0 + \mu p) V d\Omega_f \]
\[ \dot{W} = K_W p^n V^m \]
\[ J_g = \int_{\Omega_f} (p - p_g)^2 d\Omega_f \]

The total force of the contact pressure
The total moment
Power dissipation caused by friction
The wear rate of the material
The functional differences
Problem Formalization of Multipurpose Punch Shape Optimization

Find Function

\[ f^*(x, y) : J^* = J(f^*) = \min_J J(f) \]

under condition

\[ p \in \Lambda_p = \{ p(x, y) \geq 0, \int_{\Omega_f} p(x, y) d\Omega_f = P^*, \int_{\Omega_f} xp(x, y) d\Omega_f = M^* \}, P^* > 0, M^* > 0 \]

Here min-operator consider in Pareto-sence

\[ f^*_p = \arg \min_J J(f) \]
Problem 1 (Finding \( p_*(x,y) \))

\[
J = \{J_g, J_F, J_W\}^T \rightarrow \min_{p \in \Lambda_p} \left\{ p(x,y) \geq 0, \int_{\Omega_f} p(x,y)d\Omega_f = P_*, \int_{\Omega_f} xp(x,y)d\Omega_f = M_* \right\}, P_* > 0, M_* > 0
\]

Problem 2 (Finding \( f_*(x,y) \))

\[
p_*(x,y) \Rightarrow w_*(x,y)
f_*(x,y) = w_*(x,y)
\]
Optimal distribution for a rectangular stamp

Optimal Pressure Distribution

\[ p^*(x, y) = \frac{P^*}{S} + \frac{M^*}{I_y} x \]

Optimal Punch Shape

\[
\begin{align*}
  f^*(x, y) &= \kappa_0 (C + Ax) D_0(x, y) + \kappa_0 A D_C(x, y) / 2 - \\
  & - \mu \kappa_f \left[ \left( C + Ax \right) D_{0C}(x, y) + A D_{CC}(x, y) / 2 \right] - \tau_0 \kappa_f D_{0C}(x, y) \\
  C &= \frac{P^*}{S}, \quad A = \frac{M^*}{I_y}, \quad \kappa_0 = \frac{1-\nu^2}{\pi E}, \quad \kappa_f = \frac{(1+\nu)(1-2\nu)}{2\pi E}
\end{align*}
\]
Shape Functions for Rectangular Shape
Pareto Optimization Front Construction

\[ J_g = \frac{1}{\kappa_2} (\kappa_1 - J_F)^2 \]

\[ \kappa_1 = \mu V \int_{\Omega_f} p_g d\Omega_f \]

\[ \kappa_2 = \mu^2 V^2 S \]

\[ S = meas \Omega_f \]
Optimal Rectangular Punch Shape

Some Examples

Solution

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Multipurpose Optimization with and without Uncertainties for Deformed Bodies and Structural Elements
Dependence on Coefficient $C_g$
Multipurpose minimization of horizontal and vertical displacements of a free end of console beam

A.V. Sinitsin, S.Yu. Ivanova, E.V. Makeev, and N.V. Banichuk
Multipurpose minimization of horizontal and vertical displacements of a free end of console beam

\[ J_1 = u(L) = q_y^0 \int_0^L \frac{(x - L)^2}{D_1} \, dx, \quad J_2 = v(L) = q_z^0 \int_0^L \frac{(x - L)^2}{D_2} \, dx \]

\[ J = J(h_1, h_2) = \left\{ \begin{array}{ll} J_1(h_1, h_2) \\ J_2(h_1, h_2) \end{array} \right\} \rightarrow \min_{h_1, h_2 \in \Lambda_h} \]

\[ h \in \Lambda_h = \{h_1, h_2 : h_1(x) > 0, h_2(x) > 0, 0 \leq x < L, M(h_1, h_2) = M_0\} \]
Multipurpose minimization of horizontal and vertical displacements of a free end of console beam

\[ J_1^P = \frac{P}{J_2^P} \]

\[ J_1^N = \frac{N}{J_2^N} \]


THANK YOU FOR YOUR ATTENTION!